Commun. Combin., Cryptogr. & Computer Sci., 1 (2022), 55–61

ISSN:2783-5456

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: http://cccs.sgh.ac.ir

An Accurate Approach to The Simulation of Nonlinear Generalized Fractional Fisher Equation

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Abstract

The Fisher dissemination exchange properties, as well as reaction from characteristics, make the non-linear model. The nonlinear fractional Fisher model shows up in practical physical circumstances like ultra-slow kinetics, Brownian movement of particles, anomalous diffusion, polymerases of Ribonucleic acid, deoxyribonucleic acid, continuous irregular activity, and arrangement of wave kinds. The paper considered the strategy based on the Chebyshev polynomials to get the numerical method to solve the nonlinear generalized fractional Fisher equation. The numerical scheme is developed in the following manners: at first, the semi-discrete is constructed in the temporal sense based on a linear interpolation with accuracy order $\delta^2 t$, and secondly, the full discrete of the model is investigated. Moreover, the unconditional stability and convergence order are investigated via the numerical results. For getting of the full-discrete scheme, the spatial derivative is approched based on the shifted Chebyshev basis. In addition, the adequacy and legitimacy of the proposed modern are illustrated by means of two test.

Keywords: Fisher model, Nonlinear generalized fractional Fisher equation, Chebyshev polynomials, Collocation method.

2020 MSC: 34K37, 97N50.

1. Introduction

In recent papers, the fractional model was proposed for modeling the research of many researchers such as mathematics, physics, and engineering, they have ideally studied fractional calculus because of its wide applications [1, 2, 3]. The most common definitions used in the fractional partial equation are Caputo derivatives that defined as

$$ {}_0\mathcal{D}_x^{\upsilon} u(x,t) = \frac{1}{\Gamma(\Theta-\upsilon)} \int_a^x (x-\zeta)^{\Theta-\upsilon-1} \frac{\partial^{\Theta} u(x,t)}{\partial \zeta^{\Theta}} d\zeta. $$

where ${}_0\mathcal{D}_x^{\alpha} u(x,t)$ denotes the left Caputo fractional derivative of $u(x,t)$ of the order $\Theta-1 < \alpha \leqslant \Theta$, $\Theta \in \mathbb{N}$. Nonlinear fractional partial differential equations have attained substantial importance. This paper is to examine and estimate a solution for the general form of the nonlinear generalized fractional Fisher equation (NGFFE) using the Caputo fractional differential analysis as

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Received: November 3, 2022 Revised: November 10, 2022 Accepted: November 21, 2022

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{D}_x^{\alpha} u(x,t) + u(1-u)(u-\zeta) + f(x,t), \qquad 0 \leqslant x \leqslant L,\ 0 \leqslant t \leqslant T, \tag{1.1}$$

where $0 < \zeta < 1$. The constant function $f(x,t)$ has been added to the model so that the accuracy of the numerical method can be conveyed to the reader in a clearer way. The initial condition is

$$u(x,0) = q(x), \qquad 0 \leqslant x \leqslant L, \tag{1.2}$$

and boundary conditions are the following.

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \qquad 0 \leqslant t \leqslant T, \tag{1.3}$$

${}_0\mathcal{D}_x^{\alpha} u(x,t)$ denotes the left Caputo fractional derivative of $u(x,t)$ of the order $\Theta - 1 < \alpha \leqslant \Theta$, $\Theta \in \mathbb{N}$. Some of the applications of this equation are control theory, Nano-electrodynamics, Neurophysiology, and autocatalytic chemical reaction [4, 5].

To solve NGFFE, numerical approaches must be used in order to gain an approximate solution. Because the analytical solution to solve this type of problem is difficult and in many cases, the answer can be reached with limited conditions, otherwise, it is impossible. During the past few years, many authors presented some numerical methods, for example, Fisher's nonlinear diffusion formulas with general solutions for fractional sector [6], Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative [7], utilizing the fractional sub-equation approach with analytical solutions to the fractional Fisher equation [8], on the time fractional generalized Fisher equation [9], a numerical investigation that solves a fractional Fisher problem by using Chebyshev transformation algorithm [10].

However, these schemes require more or less some restrictions on the mesh ratio, i.e., they are conditionally stable. But in this article, we will present an unconditionally stable numerical method. The goals of this paper are as follows. In part 2, we define the explanation of the temporal and spatial discretization, shifted Chebyshev polynomials, and some advantages. In the next part of the paper, the numerical method is explained to approximate the nonlinear problem (1.1). The numerical results, error estimates, tables, and different figures are demonstrated in Section 3.

2. Transposed Chebyshev polynomials of the first type

This paper segment shows how to use the finite difference method in the temporal sense. Then we use the shifted Chebyshev base in the collocation technique to discrete the space variable for Eq. (1.1). Primary subsections of the Chebyshev polynomials of the first type (CPFT) will be introduced in particularity as follow.

Firstly, CPFT is created based on the Jacobi polynomials $\mathcal{P}_k^{(\eta,\theta)}(x)$ as below

$$\mathcal{T}_k(x) = \frac{2^{2k}}{\binom{2k}{k}} \mathcal{P}_k^{(-\frac{1}{2},-\frac{1}{2})}(x).$$

The closed form of $\mathcal{T}_k^{Ⓢ}(x)$ in the distance $x \in [0,1]$ can be obtained as

$$\mathcal{T}_k^{Ⓢ}(x) = \sum_{\ell=0}^{k} \mathcal{A}_{k,\ell} \times x^{k-\ell}, \qquad k = 1, 2, ..., M+1, \tag{2.1}$$

where

$$\mathcal{A}_{k,\ell} = \frac{(-1)^{\ell} \times 2^{2k-2\ell} k \Gamma(2k-\ell)}{\Gamma(\ell+1) \times \Gamma(2k-2\ell+1)},$$

and $\mathcal{T}_k^{Ⓢ}(x)$ is the shifted Chebyshev polynomials of the first type (SCPFT). Accordingly we derive that $\mathcal{T}_k^{Ⓢ}(x)$ are orthogonal polynomials on $[0,1]$ with the inner product that is

$$\langle \mathcal{T}_k^{Ⓢ}(x), \mathcal{T}_n^{Ⓢ}(x)\rangle = \int_0^1 \frac{1}{\sqrt{x-x^2}}\ \mathcal{T}_k^{Ⓢ}(x)\ \mathcal{T}_n^{Ⓢ}(x)\ dx = \begin{cases} \frac{\pi}{2}, & k=n=0, \\ \pi, & k=n, \\ 0, & k \neq n. \end{cases}.$$

Now we may approximate the $u(x,t)$ on $(x,t) \in [0,1]\times[0,T]$ by $\mathcal{T}_k^{Ⓢ}(x),\ \ k=0,1,\dots,M$ as

$$u(x,t) \approx u^*(x,t) = \sum_{k=0}^{M} \rho_k(t)\ \mathcal{T}_k^{Ⓢ}(x) = \Upsilon\ \mathcal{T}^{Ⓢ^T}, \tag{2.2}$$

in which

$$\rho_k = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x-x^2}}\ u^*(x,t)\ \mathcal{T}_k^{Ⓢ}(x)\ dx.$$

In Eq. (2.2), Υ and $\mathcal{T}^{Ⓢ}$ are the vector in $(M+1)$-dimensions that specified as

$$\mathcal{T}^{Ⓢ} = \left[\mathcal{T}_0^{Ⓢ}(x), \mathcal{T}_1^{Ⓢ}(x), \dots, \mathcal{T}_{M+1}^{Ⓢ}(x)\right], \qquad \Upsilon = [\rho_0, \rho_1, \dots, \rho_{M+1}],$$

In arrange to realize the numerical strategy, we need the Caputo fractional derivative $\mathcal{T}_k^{Ⓢ}(x)$ in Eq. (2.1) that determine as

$${}_0\mathfrak{D}_x^{\alpha}(\mathcal{T}_k^{Ⓢ}(x)) = \sum_{\ell=0}^{k-\lceil\alpha\rceil} \mathfrak{L}_{k,\ell}^{\alpha,\lceil\alpha\rceil}\ x^{k-\ell-\alpha}, \qquad x\in[0,1], \quad k=1,2,\dots,$$

where

$$\mathfrak{L}_{k,\ell}^{\alpha,\lceil\alpha\rceil} = \frac{(-1)^{\ell}4^{k-\ell}k\Gamma(2k-\ell)\Gamma(k-\ell+1)}{\Gamma(\ell+1)\Gamma(2k-2\ell+1)\Gamma(k+1-\ell-\lceil\alpha\rceil)}.$$

Consider that for $k < \lceil\alpha\rceil$ we have ${}_0\mathfrak{D}_x^{\alpha}(\mathcal{T}_k^{Ⓢ}(x)) = 0$. Then the α-order of the Caputo derivative of the function $u(x,t) \in C([0,1]\times[0,T])$ can be obtained by the next formula.

$${}_0\mathfrak{D}_x^{\alpha}(u^*(x,t)) = \sum_{k=0}^{M}\rho_k(t){}_0\mathfrak{D}_x^{\alpha}(\mathcal{T}_k^{Ⓢ}(x)) = \sum_{k=0}^{M}\sum_{\ell=0}^{k-\lceil\alpha\rceil} \rho_k(t)\ \mathfrak{L}_{k,\ell}^{\alpha,\lceil\alpha\rceil}\ x^{k-\ell-\alpha} = \Upsilon\ {}_{\alpha}\mathcal{T}^{Ⓢ^T}, \tag{2.3}$$

where ${}_{\alpha}\mathcal{T}^{Ⓢ} = \left[{}_{\alpha}\mathcal{T}_0^{Ⓢ}(x), {}_{\alpha}\mathcal{T}_1^{Ⓢ}(x), \dots, {}_{\alpha}\mathcal{T}_{M+1}^{Ⓢ}(x)\right]$ is a vector in $(M+1)$-dimensions that determine as

$${}_{\alpha}\mathcal{T}_k^{Ⓢ}(x) = \sum_{\ell=0}^{k-\lceil\alpha\rceil} \mathfrak{L}_{k,\ell}^{\alpha,\lceil\alpha\rceil}\ x^{k-\ell-\alpha}, \qquad k=1,2,\dots,M.$$

2.1. Manner of the numerical scheme

This segment is committed to combining the spectral strategy based on SCPFT with the finite difference to generate the numerical scheme of Eq. (1.1). Imprimis, the integer-order temporal derivative of Eq. (1.1) is discretized using the following structure as

$$\frac{\partial u(x,t)}{\partial t} = \frac{u(x,t_{z+1})-u(x,t_z)}{\delta t} - \frac{\delta t}{2}\frac{\partial^2 u(x,t)}{\partial^2 t} + O(\delta^2 t), \qquad z=1,2,\dots,N_t, \tag{2.4}$$

in which N_t is the measure of step of the temporal variable t and $\delta t = \frac{T}{N_t}$. To get the temporal-discrete method with the second order, replace (2.4) in Eq. (1.1) and regiment it, one obtain as

$$u^{z+1} - \frac{1}{2}\delta t \ {}_0\mathfrak{D}_x^\alpha u^{z+1} - \frac{\delta t}{2} \ N(u^{z+1}) = u^z + \frac{1}{2}\delta t \ {}_0\mathfrak{D}_x^\alpha u^z + \frac{\delta t}{2} \ N(u^z) + \frac{1}{2}(f^{z+1} + f^z) + \delta t \ R^{z+1}, \tag{2.5}$$

where $N(u^z) = u^z(1-u^z)(u^z-\zeta)$. In the past relation, u^z and f^z are explaining $u(x,t_z)$ and $f(x,t_z)$, respectively. As well as, $R^{z+1} = O(\delta^2 t)$ is the truncation error. The temporal scheme is got by deleting the truncation error as

$$U^{z+1} - \frac{1}{2}\delta t \ {}_0\mathfrak{D}_x^\alpha U^{z+1} - \frac{\delta t}{2} \ N(U^{z+1}) - U^z - \frac{1}{2}\delta t \ {}_0\mathfrak{D}_x^\alpha U^z - \frac{\delta t}{2} \ N(U^z) - \frac{1}{2}(f^{z+1} + f^z) = 0, \tag{2.6}$$

where V^z is the approximate solution. At present, permit us to create the spatial discretization, replacing (2.2) and (2.3) in (2.6), we get

$$\Upsilon^{z+1}\left[\mathcal{C}^{\circledS^T} - \frac{1}{2} \ \delta t \ {}_\alpha\mathcal{C}^{\circledS^T}\right] - \frac{\delta t}{2} \ N(\Upsilon^{z+1} \ \mathcal{C}^{\circledS^T}) - \Upsilon^z\left[\ \mathcal{C}^{\circledS^T} + \frac{1}{2}\delta t \ {}_\alpha\mathcal{C}^{\circledS^T}\right] - \frac{\delta t}{2} \ N(\Upsilon^z \ \mathcal{C}^{\circledS^T}) - \frac{1}{2}(f^{z+1} + f^z) = 0. \tag{2.7}$$

As a result, we dispose Eq. (2.7) by using $M+1$ points $\{x_r\}_{r=0}^{r=M}$ on $x \in (0,1)$ that the roots of SCPFT are as these points then we get a method of nonlinear system of $M-1$ equations and $M+1$ unknowns. To get the other two equations, we apply the boundary conditions that are gained by substituting Eq. (2.2) in Eq. (1.3) as

$$\begin{cases} g_0^z = \sum_{k=0}^{M} \rho_k^z \ \mathcal{T}_k^{\circledS}(0), \\ g_1^z = \sum_{k=0}^{M} \rho_k^z \ \mathcal{T}_k^{\circledS}(1), \end{cases} \qquad z = 0, 1, \ldots, N_t, \tag{2.8}$$

where g_0^z and g_1^z are shortened $g_0(t_z)$ and $g_1(t_z)$.
The nonlinear system (2.7)-(2.8) can be present with a form as

$$\mathcal{H}(\rho_k^z) = 0, \qquad z = 0, 1, \ldots, N_t, \tag{2.9}$$

One can construct the nonlinear system (2.9) by using the Newton iteration approach in the below formulation.

$$\breve{\rho}_k(t_{z+1}) = \breve{\rho}_k(t_z) - \mathfrak{J}^{-1}(\breve{\rho}_k(t_z)) \ \mathcal{H}(\breve{\rho}_k(t_z)),$$

where $\mathfrak{J}^{-1}(\breve{\rho}_k(t_z))$ is the inverse of the Jacobian matrix. For $j = 0$, we use the initial condition as

$$u(x,0) \approx \sum_{k=0}^{M} \rho_k(0) \ \mathcal{T}_k^{\circledS}(x) = q(x),$$

where $\rho_k(0) = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x-x^2}} \ q(x) \ \mathcal{T}_k^{\circledS}(x) \ dx$. As a result, we can obtain unknowns $\rho_k(t)$.

3. Test cases to approve the numerical method

In this segment, the numerical consideration of distinctive test cases to look at the adequacy and validity is reported by considering the proposed strategy in terms of errors $\mathcal{L}_\infty$ and $\mathcal{L}_2$-error as

$$\mathcal{L}_2 = \left(\frac{1}{M}\sum_{r=0}^{M} | \ u^*(x_r,T) - u(x_r,T) \ |^2\right)^{\frac{1}{2}},$$
$$\mathcal{L}_\infty = \max | \ u^*(x_r,T) - u(x_r,T) \ |, \quad 0 \leqslant r \leqslant M.$$

The numerical come about of the procedure upon a few test issues for different values of N_t and M are utilized to assess the accuracy and consistency of the utilized scheme. Notice that to estimate the temporal convergence order $\mathcal{C}_{order}$, we use the following rule.

$$\mathcal{C}_{order} = \log_2\left(\frac{error(N_t,M)}{error(2N_t,M)}\right).$$

Problem 3.1. Consider the test example as

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{D}_x^{\alpha} u(x,t) + u(x,t)(1-u(x,t))(u(x,t)-\zeta) + f(x,t), \qquad 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T,$$

in which

$$f(x,t) = -e^{-t}\left(x^2 + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}\right) - e^{-t}\left(x^2(1-x^2)(x^2-\zeta)\right).$$

The analytical solution is $u(x,t) = x^2 \exp(-t)$. The set of numerical results of this example is presented in tables and figures as follows. Table 1 shows the order with $\alpha = 0.5$ at $T = 1$, and $M = 6$ and different values of N_t. As to be shown, the produced order is in total agreement with the hypothetical order in this table. In Table 2, $\mathcal{L}_\infty$ and $\mathcal{L}_2$ are compared for $M = 7, 8, 9$ with value $N_t = 100$ at $T = 1$ and $\alpha = 0.6$. In Figure 1, the absolute error for eight collocation points and the various values M at $T = 1$ and $\alpha = 0.8$ is presented. These results indicate the proposed method is also very efficient from the computational standpoint.

Table 1: The result with $\alpha = 0.5$, $M = 6$ and different values of N_t at $T = 1$ for Example 3.1.

N_t	$\alpha = 0.5$			
	$\mathcal{L}_\infty$	C_{order}	$\mathcal{L}_2$	C_{order}
15	3.67930×10^{-2}	–	7.20027×10^{-2}	
30	5.48141×10^{-3}	2.74681	1.06776×10^{-2}	2.75346
60	1.21866×10^{-3}	2.16925	2.37020×10^{-3}	2.17151
120	2.96054×10^{-4}	2.04136	5.75566×10^{-4}	2.04196
240	7.34869×10^{-5}	2.01030	1.42853×10^{-4}	2.01045

Table 2: The comparison of $\mathcal{L}_\infty$ and $\mathcal{L}_2$ for $M = 7, 8, 9$ and $N_t = 100$ at $T = 1$ for Example 3.1.

M	$\alpha = 0.6$	
	$\mathcal{L}_\infty$	$\mathcal{L}_2$
7	3.21860×10^{-4}	4.26890×10^{-4}
8	2.4860×10^{-4}	5.78203×10^{-4}
9	9.25482×10^{-5}	4.86102×10^{-4}

Problem 3.2. Consider the test example as below

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{D}_x^{\alpha} u(x,t) + u(x,t)(1-u(x,t))(u(x,t)-\zeta) + f(x,t),$$
$$0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant 1,\ 0 < \alpha \leqslant 2.$$

where

$$f(x,t) = -\exp(-t)\Bigg(x^2(1-x) + \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} - \frac{\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha} + x^2(1-x)(1-x^2(1-x))(x^2(1-x)-\zeta)\Bigg).$$

The initial and boundary conditions are as below, respectively.

$$u(x,0) = x^2(1-x), \quad u(0,t) = u(1,t) = 0.$$

The analytic solution of this example is $u(x,t) = x^2(1-x)\exp(-t)$.

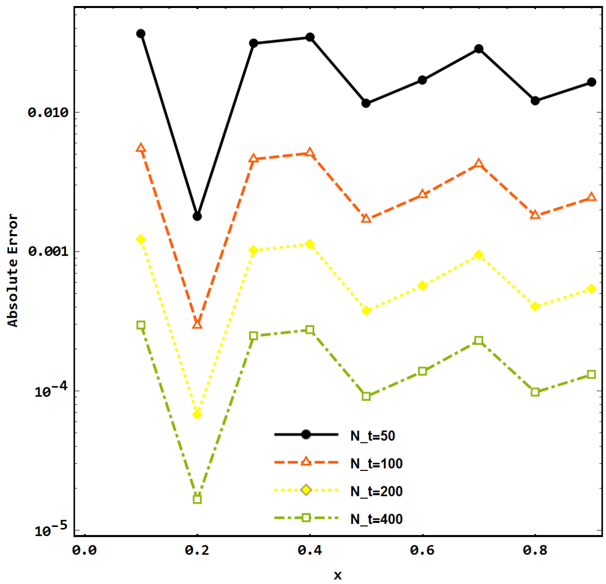

Figure 1: The error $\mathcal{L}_\infty$ with $\alpha = 0.8$ at $T = 1, M = 6$ for Example 3.1.

The set of numerical results of this example is presented in tables and figures as follows. Table 3 shows the order with $\alpha = 1.5$ at $T = 1$, and $M = 5$ and different values of N_t. As to be shown, the produced order is in total agreement with the hypothetical order in this table. In Table 4, $\mathcal{L}_\infty$ and $\mathcal{L}_2$ are compared for $M = 4, 5, 6$ with value $N_t = 300$ at $T = 1$ and $\alpha = 1.7$. In Figure 2, the absolute error for eight collocation points and the various values M at $T = 1$ and $\alpha = 0.5$ is presented. These results indicate the proposed method is also very efficient from the computational standpoint.

Table 3: The result with $\alpha = 1.5$, $M = 5$ and different values of N_t at $T = 1$ for Example 3.2.

N_t	$\alpha = 1.5$			
	$\mathcal{L}_\infty$	C_{order}	$\mathcal{L}_2$	C_{order}
15	3.08007×10^{-3}	–	6.74625×10^{-3}	
30	7.70318×10^{-4}	1.99944	1.68724×10^{-3}	1.99942
60	1.92599×10^{-4}	1.99986	4.21854×10^{-4}	1.99985
120	4.8151×10^{-5}	1.99996	1.05466×10^{-4}	1.99996
240	1.20378×10^{-5}	1.99999	2.63667×10^{-5}	1.99999

Table 4: The comparison of $\mathcal{L}_\infty$ and $\mathcal{L}_2$ for $M = 4, 5, 6$ and $N_t = 300$ at $T = 1$ for Example 3.2.

M	$\alpha = 1.2$	
	$\mathcal{L}_\infty$	$\mathcal{L}_2$
4	4.72802×10^{-3}	8.35482×10^{-2}
5	4.70150×10^{-3}	8.00189×10^{-2}
6	3.48902×10^{-3}	3.07821×10^{-2}

4. Conclusion

After all, the paper persuades the readers that this is the simplest method to solve the nonlinear generalized fractional Fisher equation with the numerical scheme. The last step in the reasoning scheme of

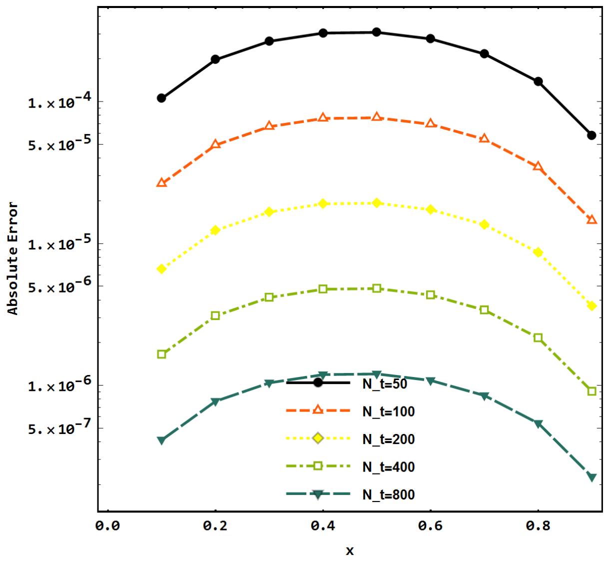

Figure 2: The error $\mathcal{L}_\infty$ with $\alpha = 1.7$ at $T = 1, M = 8$ for Example 3.2.

the paper is its convergence and stability. This strategy is based on the finite difference to generate the semi-discrete and the collocation method on the basis of the Chebyshev polynomials to get the full discrete. The current method is very accurate and its efficiency is simpler and more acceptable than other methods. This method can be used for many other non-linear equations that have a fractional derivative with respect to time and space variables.

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